Chapter 6 – Probability

Learning Objectives

1. Simulate simple experiments and compute empirical probabilities.
2. Compute both theoretical and empirical probabilities.
3. Apply the rules of probability to compute probabilities.
4. Compute probability based on odds and compute odds based on probability.

Simulation:

Simulating tossing two coins with your calculator.

1. Clear three lists on your TI83/84: L1, L2, L3
2. Code the data: let 1 = head, 0 = tail
3. Generate n random integers in L1 and L2: \text{randInt}(0,1,n)
4. Go up to L3 and enter L1 + L2
5. Produce a bar graph
Empirical Probability

Notation:

Theoretical Probability

Notation:

Sample Space

1. Set of all possible outcomes in an experiment.
2. A simple sample space is one in which all possible outcomes have equal probability.
Event

Consider tossing three coins. Let A be the event “observed two heads.” List the sample space.

\[ P(A) = \frac{n(A)}{n(S)} \]

Consider actually tossing a single coin 10 times. \( \hat{P}(\text{heads}) = \frac{n(H)}{n} \)

Probabilities ALWAYS reside on \([0, 1]\)

Law of Large Numbers
\( \hat{P}(A) \rightarrow P(A) \)
The Odds of an Event is defined as:

Mathematically, this is represented as: \( \frac{\pi}{1 - \pi} \)

**Example:** The probability you will observe only one head when tossing three coins is (enumerate the sample space and figure it out):

This means the odds of observing only one head is then:

**Example:** Given the odds of a particular team winning a tournament is 3:5, what is the probability they win?
The Compliment of an Event

The complement of an event is defined as:

Notation: Given an event labeled as $A$, the compliment of $A$ is denoted as:

Consider the following table, which is found on page 102, section 6.8.1, of your text book.

<table>
<thead>
<tr>
<th>Hours Employed</th>
<th>Freshman</th>
<th>Sophomore</th>
<th>Junior</th>
<th>Senior</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$hrs \leq 5$</td>
<td>24</td>
<td>18</td>
<td>13</td>
<td>8</td>
<td>63</td>
</tr>
<tr>
<td>$5 &lt; hrs \leq 10$</td>
<td>12</td>
<td>17</td>
<td>15</td>
<td>19</td>
<td>63</td>
</tr>
<tr>
<td>$10 &lt; hrs \leq 20$</td>
<td>13</td>
<td>10</td>
<td>12</td>
<td>15</td>
<td>50</td>
</tr>
<tr>
<td>$20 &lt; hrs$</td>
<td>17</td>
<td>11</td>
<td>31</td>
<td>45</td>
<td>104</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>66</strong></td>
<td><strong>56</strong></td>
<td><strong>71</strong></td>
<td><strong>87</strong></td>
<td><strong>280</strong></td>
</tr>
</tbody>
</table>
Mutually Exclusive Events

Two events are said to be Mutually Exclusive if:

Consider the table of Hours Employed by Class Standing table once again.

<table>
<thead>
<tr>
<th>Hours Employed</th>
<th>Class Standing</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Freshman</td>
</tr>
<tr>
<td>hrs ≤ 5</td>
<td>24</td>
</tr>
<tr>
<td>5 &lt; hrs ≤ 10</td>
<td>12</td>
</tr>
<tr>
<td>10 &lt; hrs ≤ 20</td>
<td>13</td>
</tr>
<tr>
<td>20 &lt; hrs</td>
<td>17</td>
</tr>
<tr>
<td>Total</td>
<td>66</td>
</tr>
</tbody>
</table>

Given the event “Freshman”, identify:

- One event that is mutually exclusive and explain why it is mutually exclusive.

- One event that is not mutually exclusive and explain why it is not mutually exclusive.
Conditional Probability

The basic idea behind Conditional Probability is that it is a probability calculated based on knowing another event (the condition) has occurred.

Given two events, A and B, the conditional probability of A “given” B is denoted as:

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

Example:
1. Find the probability of a freshman.
2. Find the probability of a freshman given the student worked more than 20 hours.

<table>
<thead>
<tr>
<th>Hours Employed</th>
<th>Freshman</th>
<th>Sophomore</th>
<th>Junior</th>
<th>Senior</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>hrs ≤ 5</td>
<td>24</td>
<td>18</td>
<td>13</td>
<td>8</td>
<td>63</td>
</tr>
<tr>
<td>5 &lt; hrs ≤ 10</td>
<td>12</td>
<td>17</td>
<td>15</td>
<td>19</td>
<td>63</td>
</tr>
<tr>
<td>10 &lt; hrs ≤ 20</td>
<td>13</td>
<td>10</td>
<td>12</td>
<td>15</td>
<td>50</td>
</tr>
<tr>
<td>20 &lt; hrs</td>
<td>17</td>
<td>11</td>
<td>31</td>
<td>45</td>
<td>104</td>
</tr>
<tr>
<td>Total</td>
<td>66</td>
<td>56</td>
<td>71</td>
<td>87</td>
<td>280</td>
</tr>
</tbody>
</table>
Join Probabilities

The previous statement \( P(A \mid B) = \frac{n(A \cap B)}{n(B)} \) includes the idea of joint probability.

\[
\hat{P}(\text{Sophomore} \cap \text{hrs} \leq 5) = \frac{n(S \cap hrs \leq 5)}{n} = 
\]

**Independence:** Two events are said to be independent if the occurrence of one event does not affect the probability of the other event occurring.

**Some Examples:**
Consider two arbitrary events, A and B. If A and B are independent then the fact that B just occurred will not have any affect on the probability of A occurring.

\[
P(A | B) = \frac{P(A \cap B)}{P(B)} \quad \therefore \text{therefore}
\]

Go back to the table. Are the events Freshman and More than 20 Independent? Show mathematically why.
School administrators are proposing a change in the academic calendar. Sixty students were surveyed and asked if they preferred (A) a shorter Christmas break by 1 week which would result in a longer summer break by 1 week, or (B) a longer Christmas break by 1 week which would result in a shorter summer break by one week. The results are shown below.

<table>
<thead>
<tr>
<th></th>
<th>Fresh</th>
<th>Soph</th>
<th>Jr.</th>
<th>Sr.</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>4</td>
<td>7</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>B</td>
<td>10</td>
<td>11</td>
<td>14</td>
<td>5</td>
<td>40</td>
</tr>
<tr>
<td>Total</td>
<td>15</td>
<td>15</td>
<td>21</td>
<td>9</td>
<td>60</td>
</tr>
</tbody>
</table>

Are the events Sr. and option A Independent?

Are the events Freshman and option A Independent?
The General Addition Rule:  \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \)

The General Multiplication Rule:  \( P(A \cap B) = P(B)P(A \mid B) \)

For the special case of independence this simplifies to:

Is Independence and Mutually Exclusive the Same Thing?
A few practice problems:

Given the following, \( P(A) = 0.13 \), \( P(B) = 0.27 \), \( P(C) = 0.18 \), \( P(D) = 0.62 \), the events A and D are mutually exclusive, the events B and C are independent, find:
Chapter 7
Random Variables and Probability Distributions

Learning Objectives:

1. Be able to identify a discrete and continuous random variable.
2. Construct a probability distribution for a discrete random variable.
3. Compute the mean, variance and standard deviation for a discrete random variable.

Random Variable: A variable that is assigned to an item of interest in a statistical study such that the values it can assume are the result of chance or a random event.

Example:

Notation:

Discrete vs. Continuous Random Variables
Probability Distribution of a Discrete Random Variable
A table or function that lists all of the possible outcomes and associated probabilities of a discrete random variable.

A probability distribution is a theoretical distribution. It is associated with population probabilities.

Properties of a proper Probability Distribution

1. All probabilities are on \([0, 1]\)
2. Sum of all probabilities over the entire sample space is 1.

Example: Consider tossing 3 coins and recording the number of heads.

1. List all possible outcomes
2. List the probabilities associated with each outcome
Mean of a Discrete Distribution

\[ \mu = \sum_{i=1}^{N} x_i P(X = x_i) \]

This is just a weighted average, where each outcome is weighted by the probability of observing that outcome. As such, outcomes with higher probability will be weighted heavier and have more influence on the value.

For the coin toss example this would be:
The equation for the variance is:

For the coin toss example this would be:
Chapter 8
The Binomial Probability Distribution

Learning Objectives:

• Identify an experiment as being binomial, or not.
• Compute probabilities associated with events of a binomial distribution.
• Compute the mean, variance and standard deviation of a binomial random variable.

• *First named distribution. There will be many more.*
• *Why are probability distributions important?*
Properties of a Binomial Probability Experiment

1. There are n repeated, independent and identical trials.

2. Only two possible outcomes for each trial, which we label a success or failure.

3. $P(\text{Success})$ remains constant for all trials.

4. The random variable records the number of success in n trials.

Data Types: nominal

Why?
Example: Michelle steps to the free throw line to take 13 shots. Historically she has been hitting 78.3% from the free throw line. We want to calculate various probabilities regarding how many successful shots she will have. Can we use a binomial distribution to reasonably calculate those probabilities? Why?

Example: Michelle steps to the free throw line to take 13 shots. Historically she has been hitting 78.3% from the free throw line.

Find: \( P(\text{She hits exactly 9}) \)

\[
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \\
\]

\text{binompdf:}

\text{binomcdf:}
Find: \( P(\text{She hits less than 9}) \)

\[
\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
\]

Find \( P(\text{She hits more than 9}) \)

\[
\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
\]

Find \( P(\text{She hits from 4 to 9}) \)

\[
\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
\]

Find \( P(\text{She hits between 3 and 11}) \)

\[
\begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10
\end{array}
\]
The mean of the binomial distribution: $\mu = n\pi$
Find the mean for Michelle’s scenario.

The variance of the binomial distribution: $\sigma^2 = n\pi(1 - \pi)$
Find the variance for Michelle’s scenario.

This implies the standard deviation must be:

Find the standard deviation for Michelle’s scenario.
Chapter 9
The Normal Distribution

Learning Objectives:

- Convert values of a non-standard normal random variable to values of a standard normal random variable.
- Compute probabilities associated with both standard normal and non-standard normal random variables.
- Compute percentiles for normal random variables.

Properties of a Normal Distribution

- The normal distribution is our first _________________ distribution in comparison to the binomial distribution which is ________________.
- The normal distribution is a bell shaped curve.
- The total area under the curve is __________.
- The curve extends to ________________.
- The distribution is centered at the _____________. Since the distribution is bell shaped, thus symmetric, the ____________ and ____________ have the same value.
- Probabilities are associated with area under the curve. The probability a value is observed between any two points in the distribution is equal to the area under the curve between those two points.
The Standard Normal Distribution

- The standard normal distribution is a special normal distribution in that the mean is equal to ________ and the variance is equal to ________.

- The Empirical Rule is based on the Standard Normal Distribution.

- Any normal distribution that has a mean other than ________ or a variance other than ________ is called a Non-standard Normal Distribution.

Finding Probabilities

Example: Wine manufacturers are very cognizant of the extraction force required to open a corked bottle of wine. It is considered an industry standard for the average force to be 305 Newtons with a standard deviation of 28 Newtons. A bottle requiring less than 240 Newtons will not be sealed tight enough and a bottle requiring more than 378 Newtons will be too tight for most consumers to comfortably open. One Newton = 1 kilogram meter/sec². Assuming the distribution of force is distributed normally, find the following probabilities.

Notation:

1. Find the probability a randomly selected bottle of wine will require too much force to open than would be considered comfortable for the average consumer.

   Probability Statement:

   Draw a Picture:
2. Find the probability a randomly selected bottle of wine will not be sealed tightly enough.

**Probability Statement:**

**Draw a Picture:**

3. Find the probability a randomly selected bottle of wine will require exactly 320 Newtons to open.

**Probability Statement:**

**Draw a Picture:**
4. Find the probability a randomly selected bottle of wine will require between 240 and 378 Newtons.

Probability Statement:

Draw a Picture:

5. Based on the information provided, what percent of wine bottles are either not sealed tight enough or are sealed too tightly?

Probability Statement:

Draw a Picture:

Repeat problem #2 by first converting values to a standard normal then computing the probability.

2. Find the probability a randomly selected bottle of wine will not be sealed tightly enough.

Use probabilities from the standard normal distribution to verify the Empirical Rule.

1. $P(-1 < z < 1)$

Draw a picture:
2. $P(-2 < z < 2)$

Draw a picture:

Percentiles

Find the 98th percentile for the force required to open a wine bottle based on $X \sim N(305, 28^2)$. 
Chapter 10
Sampling Distributions

Learning Objectives:

• Understand the similarities and differences between a population distribution and a sampling distribution.
• Use the Central Limit Theorem to describe a sampling distribution.
• Compute probabilities associated with a sampling distribution.
• Understand the relationship between sample size and the standard deviation of the sampling distribution.

The Sampling Distribution of a Sample Statistic

Concept: Consider a population of size__________. If you take all possible samples of size____
__________, calculate the sample mean from each sample, then look at the distribution of the sample mean, you have just established the_______________________________.

If we know the parameters that specify this distribution, then we can calculate probabilities based on this distribution.

Three very interesting things occur when we do this for the sample mean.

1. Regardless of the parent distribution:
   
a. The average of all the sample means will be: ____________
   
b. The variance of all the sample means will be: ____________
2. If the parent distribution (the population) in Normal, then the *sampling distribution of the sample mean* will also be normal.

\[ \bar{X} \sim N(\mu, \sigma^2) \]

3. If the sample size is large enough, then the ____________________________

will apply and we can reasonably assume:

\[ \bar{X} \sim N(\mu, \sigma^2) \]

A big enough sample size, when working with the sampling distribution of the sample mean, is typically considered to be ______________. This is just a ______________ meaning it is more of a guideline than a law carved in stone.

**The Central Limit Theorem** may be one of the most important theorems of basic statistical analysis. The reason is because:
Calculating a Probability Regarding the Mean Using the C.L.T.

The scores of 12th-grade students on the National Assessment of Educational Progress math test for the year 2000 has a distribution that is reasonably normal with a mean of 300 and a standard deviation of 35.

\[ X \sim N(300, 35) \]

1. Find the probability that a single randomly chosen student has a score of at least
   a. 300
   b. 350

2. Consider a randomly chosen group of 26 students.
   a. Describe the sampling distribution of the sample mean. Justify the parameter values you chose. \( \bar{X} \sim N(300, ) \)
b. Find the probability that a randomly chosen group of 26 students had an average score of at least 300.

c. Find the probability that a randomly chosen group of 26 students had an average score of at least 350.
The CLT and the Sampling Distribution of the Sample Proportion

The CLT also applies to the sampling distribution if the Sample Proportion. The CLT tells us that for large enough ________ the distribution of the sample proportion will be:

\[ \hat{P} \sim N\left(\pi, \frac{\pi(1-\pi)}{n}\right) \]

The requirements for applying the CLT to the sampling distribution for the sample proportions differs from that of the sample means.

1. The idea of the population being normal has no meaning.
   Why?

2. The meaning “large enough _____” also differs from that of the ________.

3. To show the sample size is ________________ for the CLT to apply, we must first show:
   a. \( n \geq 20 \)
   b. \( n\pi \geq 5 \)
   c. \( n(1-\pi) \geq 5 \)
Example: Harley-Davidson motorcycle make up approximately 14.3% of all the motor cycles registered in the United States. Suppose you were planning to conduct a survey of motorcycle owners with a sample size of 500.

1. Describe and justify the sampling distribution of the sample proportion.

2. How likely is it that at least 20% of your sample will be Harley owners?

3. Suppose you completed the survey and did find at least 20% were Harley owners.
   a. Would you consider this to be unusual? Why?
   b. What conclusions might you be able to draw from this result if it were to actually happen?